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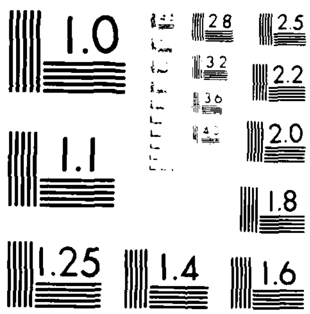

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Total Positivity: A Review

by

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Florida State University

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ABSTRACT

This paper is an invited entry for the Encyclopedia of Statistical Sciences, edited by N. L. Johnson and S. Kotz and published by John Wiley and Sons. The main objective is to review the concepts of total positivity, which plays an important role in various domains of mathematics and statistics. This article describes the power and scope of total positivity, and samples the great variety of fields of its applications.

1. Introduction.

The theory of total positivity has been extensively applied in several domains of mathematics, statistics, economics, and mechanics. In statistics, totally positive functions are fundamental in permitting characterizations of best statistical procedures for decision problems. The scope and power of this concept extend to ascertaining optimal policy for inventory and system supply problems, to clarifying the structure of stochastic processes with continuous path functions, to evaluating the reliability of coherent systems, and to understanding notions of statistical dependency.

In recent years Samuel Karlin has made brilliant contributions in developing the intrinsic relevance and significance of the concept of total

positivity to probability and to statistical theory. In 1968, Karlin wrote a classical book devoted to this vast subject. This remarkable book presents a comprehensive, detailed treatment of the analytic structure of totally positive functions, and conveys the breadth of the great variety of fields of it applications. This book, together with Karlin's other fundamental papers, inspired many new developments and discoveries in many areas of statistical applications. Frydman and Singer [8] obtained a complete solution to the embedding problem for the class of continuous-time Markov chains: The class of transition matrices for the finite state time-inhomogeneous birth and death processes coincides with the class of non-singular totally positive stochastic matrices. Keilson and Kester [21] employed total positivity to characterize a class of stochastically monotone Markov chains which has the property that the *expectation* of unimodal functions of the chain is itself unimodal in the initial state. To help unify the area of stochastic comparisons Hollander, Proschan and Sethuraman [9] introduced the concept of functions decreasing in transposition (DT). In the bivariate case, a function $f(\lambda_1, \lambda_2; x_1, x_2)$ is said to have the DT property if (a) $f(\lambda_1, \lambda_2; x_1, x_2) = f(\lambda_2, \lambda_1; x_2, x_1)$ and (b) $\lambda_1 < \lambda_2$, $x_1 < x_2$ imply that $f(\lambda_1, \lambda_2; x_1, x_2) \geq f(\lambda_1, \lambda_2; x_2, x_1)$; i.e., transposing from the natural order (x_1, x_2) to (x_2, x_1) decreases the value of the function. In their paper, total positivity is essential in showing that $P_{\underline{\lambda}}(\underline{R}=\underline{r})$, the probability of rank order $\underline{\lambda}$, is a DT function.

Karlin and Rinott [18], [19] extended the theory to multivariate cases. Multivariate total positivity properties are instrumental in [18] and [19] for the results which are applied to obtain positive dependence of random vector components and related probability inequalities.

For an excellent global view of the theory, as well as for pertinent references, the reader may consult Karlin [13].

2. Definition and Basic Properties.

2.1. Definition of Totally Positive Function.

A function $f(x,y)$ of two real variables ranging over linearly ordered one-dimensional sets X and Y, respectively, is said to be totally positive of order k (TP_k) if for all $x_1 < x_2 < \dots < x_m$, $y_1 < y_2 < \dots < y_m$ (x_i in X; y_i in Y), and all $1 \le m \le k$,

$$f\begin{pmatrix} x_1, x_2, \dots x_m \\ y_1, y_2, \dots y_m \end{pmatrix} = \begin{vmatrix} f(x_1,y_1) & f(x_1,y_2) & \dots & f(x_1,y_m) \\ f(x_2,y_1) & f(x_2,y_2) & \dots & f(x_2,y_m) \\ \vdots & \vdots & \ddots & \vdots \\ f(x_m,y_1) & f(x_m,y_2) & \dots & f(x_m,y_m) \end{vmatrix} \ge 0. \qquad (1)$$

Typically, X and Y are either intervals of the real line or a countable set of discrete values on the real line, such as the set of all integers or the set of nonnegative integers. When X or Y is a set of integers, the term "sequence" rather than "function" is used. If $f(x,y)$ is TP_k for all positive integers $k=1, 2, \dots$, then $f(x,y)$ is said to be totally positive of order ∞, written TP_∞ or TP.

A related, weaker property is that of sign regularity. A function $f(x,y)$ is sign regular of order k (SR_k) if for every $x_1 < x_2 < \dots < x_m$, $y_1 < y_2 < \dots < y_m$, and $1 \le m \le k$, the sign $f\begin{pmatrix} x_1, x_2, \dots, x_m \\ y_1, y_2, \dots, y_m \end{pmatrix}$ depends on m alone

Many well known families of density functions (both continuous and discrete) are totally positive. It should be noted that TP_2 is the order of TP-ness which has found greatest application. In the context of statistics, the TP_2 property is referred to as the monotone likelihood ratio property. Higher order TP-ness is hardly used in application except for the occasional use of TP_3.

Some examples of functions that possess the TP property are:

(i) $f(x,y) = e^{xy}$ is TP in $x,y \epsilon (-\infty,\infty)$, so that $f(x,y) = x^y$ is TP in $x \epsilon (0,\infty)$ and $y \epsilon (-\infty,\infty)$.

(ii) $f(k,t) = e^{-\lambda t}[(\lambda t)^k/k!]$ is TP in $t \epsilon (0,\infty)$ and k {0, 1, 2, ...}.

(iii) $f(x,y) = \begin{cases} 1 \text{ if } a \le x \le y \le b \\ 0 \text{ if } a \le y \le x \le b. \end{cases}$

2.2. PF_k as Special Case of Interest.

The concepts of TP_1 and TP_2 densities are familiar ones. Every density is TP_1; while the TP_2 densities are those having a monotone likelihood ratio.

A further important specialization occurs if a TP_k function may be written as a function $f(x,y) = f(x-y)$ of the difference of x and y, where x and y traverse the entire real line; f(u) is then said to be a <u>Pólya frequency function of order k</u>(PF_k). Note that a Pólya frequency function is not necessarily a probability frequency function in that $\int_{-\infty}^{\infty} f(u)du$ need not be 1 nor even finite.

The class of PF_2 functions is particularly important and has rich applications to decision theory [10], [11], [12], [18], reliability theory [3], and the stochastic theory of inventory control models [1], [16].

Every PF_2 function is of the form $e^{-\psi(x)}$, where $\psi(x)$ is convex. On the other hand, there exists no such simple representation for PF_k, $k \ge 3$. Probability densities which are PF_2 abound. For other properties and examples

of PF_2 densities, see the entry "Pólya Type 2 Frequency Distributions."

Probability densities which decrease to zero at an algebraic rate in the tails are *not* PF_2. For example, (i) Weibull with shape parameter < 1: $f(x) = \alpha\lambda(\lambda x)^{\alpha-1}\exp[-(\lambda x)^{\alpha}]$, $x \geq 0$, $\lambda > 0$, $0 < \alpha < 1$, and (ii) Cauchy: $f(x) = 1/[\pi(1+x^2)]$, $-\infty < x < \infty$ are not PF_2.

Intriguing results in the structure theory of PF_k functions can be found in Karlin and Proschan [16], Karlin, Proschan, and Barlow [17], and Barlow and Marshall [2].

2.3. Variation Diminishing Property.

An important feature of totally positive functions of finite or infinite order is their variation diminishing property: If $f(x,y)$ is TP_k and $g(y)$ changes sign at most $j \leq k-1$ times, then $h(x) \equiv \int f(x,y)\, g(y)\, dy$ changes sign at most j times; moreover, if $h(x)$ actually changes sign j times, then it must change sign in the same order as $g(y)$. It is this distinctive property which makes TP so useful. The variation diminishing property is essentially equivalent to the determinantal inequalities (1). Greater generality in stating this property is possible. The interested reader is referred to Chapter 1, Karlin [13]. A more direct approach to the theory is taken by Brown et al. [5], giving appropriate definitions and criteria for checking directly whether a family of densities possesses variation diminishing property.

2.4. Composition and Preservation Properties.

Many of the structural properties of TP_k functions are deducible from the following basic identity which is an indispensible tool in the study of total positivity.

Basic Composition Formula. Let $h(x,t) = \int f(x,y)\, g(y,t)\, d\sigma(y)$ converge abso-

lutely, where $d\sigma(y)$ is a sigma-finite measure. Then

$$h\begin{pmatrix} x_1, x_2, \ldots, x_n \\ t_1, t_2, \ldots, t_n \end{pmatrix} = \int \cdots \int_{y_1 < \cdots < y_n} f\begin{pmatrix} x_1, x_2, \ldots, x_n \\ y_1, y_2, \ldots, y_n \end{pmatrix}$$

$$\cdot g\begin{pmatrix} y_1, y_2, \ldots, y_n \\ t_1, t_2, \ldots, t_n \end{pmatrix} d\sigma(y_1) \ldots d\sigma(y_n). \qquad (2)$$

A direct consequence of the composition formula is: If $f(x,y)$ is TP_m and $g(y,t)$ is TP_n, then $h(x,t) = \int f(x,y)\, g(y,t)\, d\sigma(y)$ (the convolution of f and g) is $TP_{\min(m,n)}$. In many statistical applications this consequence is exploited principally in the case when f and g are Pólya frequency densities. That is, if $f(x)$ is PF_m and $g(x)$ is PF_n, then $h(x) = \int f(x-t)\, g(t)dt$ is $PF_{\min(m,n)}$. From this we can obtain a key result as follows.

<u>Theorem 1.</u> Let f_1, f_2, ... be density functions of nonnegative random variables with each f_i a PF_k. Then $g(n,x) = f_1 * f_2 * \ldots * f_n(x)$ (* indicates convolution) is PT_k in the variables n and x, where n ranges over 1, 2, ... and x traverses the positive real line.

The case when the random variables are not restricted to be nonnegative is discussed in Karlin and Proschan [16]. These composition and preservation properties allow us to generate other totally positive functions, thus making it easy to enlarge the TP or PF classes and to determine whether the TP property holds.

2.5. Unimodality and Smoothness Properties.

A function totally positive or more generally sign regular is endowed with certain structural properties pertaining to unimodality and smoothening properties. From the definition of PF_2 can be derived

$$\begin{vmatrix} f(x_1-y) & -f'(x_1-y) \\ f(x_2-y) & -f'(x_2-y) \end{vmatrix} \geq 0 \qquad (3)$$

for $x_1 < x_2$ and y arbitrary.

In the event that $f'(u_0) = 0$ the above inequality implies that $f'(u) \geq 0$ for $u \leq u_0$ and $f'(u) \leq 0$ for $u > u_0$. This clearly implies that if $f(u)$ is PF_2 then $f(u)$ is unimodal. In particular, every PF_2 density is a unimodal density. We note that the unimodality result is valid in case f is a PF_2 sequence.

We now describe a smoothening property possessed by the transformation under which convexity in $g(x)$ is carried over into convexity in $h(x)$, viz.

$$h(n) = \int f^{(n)}(x)g(x)dx \quad \text{for} \quad n = 1, 2, \ldots, \qquad (4)$$

where $f^{(n)}(x)$ is the n-fold convolution of f. To make this notion precise assume $f(x)$ is PF_3 and $g(x)$ is convex. Let $u = \int xf(x)dx$. Note that for arbitrary real constants a_0 and a_1,

$$\int \{g(x)-[(a_0/u)x+a_1]\} f^{(n)}(x)dx = h(n) - (a_0n+a_1). \, (5)$$

Since $g(x)$ is convex, then $g(x)-[(a_0/u)x+a_1]$ has at most 2 changes of sign and if 2 changes of sign actually occur, they occur in the order + - + as x traverses the real axis from $-\infty$ to $+\infty$. Since f is PF_3, $f^{(n)}(x)$ is TP_3 in the variables n and x by Theorem 1. The variation diminishing property implies that $h(n) - (a\ n+a_1)$ will have at most 2 changes of sign. Moreover, if $h(n) - (a_0n+a_1)$ has exactly 2 changes of sign, then these will occur in the same order as those of $g(x) - [(a_0/u)x+a_1]$, namely + - +. Since a_0, a_1 are arbitrary, we easily infer that $h(n)$ is a convex function of n. Similar results apply for concavity.

3. Applications to Statistical Decision Theory.

Historically this is perhaps the first area of statistics benefiting from the application of TP due to the great papers of Karlin [10], [11], [12]. We consider the problem of testing a null hypothesis against its alternative hypothesis, i.e., a 2-action statistical decision problem. There exist two loss functions L_1 and L_2 on the parameter space where $L_i(\theta)$ is the loss incurred if action i is taken when θ is the true parameter value. The set in which $L_1(\theta) < (>) L_2(\theta)$ is the set in which action 1 (action 2) is preferred when θ is the true state of nature. The two actions are indifferent at all other points. We shall assume that $L_1(\theta) - L_2(\theta)$ changes sign exactly n times at $\theta_1, \theta_2, \ldots, \theta_n$.

Let ϕ be a randomized decision procedure which is the probability of taking action 2 (accepting the alternative hypothesis) if x is the observed value of the random varibale X. Let C_n be the class of all monotone randomized decision procedures defined by

$$\phi(x) = \begin{cases} 1 & \text{for } x_{2i} < x < x_{2i+1},\ i = 0, 1, \ldots, [\frac{n}{2}] \\ \lambda_i & \text{for } x = x_j,\ 0 \le \lambda_j \le 1,\ j = 1, 2, \ldots, n \\ 0 & \text{elsewhere,} \end{cases} \tag{6}$$

where [a] denotes the greatest integer $\le$ a and $x_0 = -\infty$.

Using the variation diminishing property, Karlin [11] obtained the main results, which state:

<u>Theorem 2.</u> Let $f(x,\theta)$ be a strictly TP_{n+1} density and $\rho(\theta,\phi) = \int [(1 - \phi(x)) L_1(\theta) + \phi(x) L_2(\theta)] f(x,\theta) d\mu(x)$. Then for any randomized decision procedure ϕ not in C_n there exists a unique ϕ^0 such that $f(\theta,\phi^0) \le f(\theta,\phi)$ with inequalities everywhere except for $\theta = \theta_1, \theta_2, \ldots, \theta_n$.

Theorem 3. If ϕ and ψ are two procedures in C_n and f is strictly TP_{n+1} then $\int [\phi(x)-\psi(x)] f(x,\theta) d\mu(x)$ has less than n zeros counting multiplicities.

Assume $f(x,\theta)$ is strictly TP_2. For a one-sided testing problem, existence of a uniformly most powerful level α test can be easily established by Theorem 2 and Theorem 3.

More detailed discussions and other decision theoretic applications can be found in Karlin [10], [11], [12] and Karlin and Rubin [20].

4. Applications in Probability and Stochastic Processes.

Let P(t,x,E) be the transition probability funct . of a homogeneous strong Markov process whose state space is an interv on the real line and which possesses a realization in which almost all sa pl paths are continuous. Karlin and McGregor [14] *established the intimate* relationship between the general theory of TP functions and the theory of diffusion stochastic processes. Their main result shows the transition probability function P(t,x,E) is totally positive in variables x and E. That is, if $x_1 < x_2 < \ldots < x_n$ and $E_1 < E_2 < \ldots < E_n$ ($E_i < E_j$ denotes that $x < y$ for every $x \in E_i$ and $y \in E_j$), then $\det\| P(t,x_i,E_j)\| \geq 0$ for every $t > 0$ and integer n. This relation introduces the concept of a TP set function $f(x,E) = P(t,x,E)$ where t is fixed, x ranges over a subset of the real line, and E is a member of a given sigma field of sets on the line.

If the state space of the process is countably discrete, then continuity of the path functions means that in every transition of the process the particle changes "position", moving to one of its neighboring states. Thus, discrete state continuous path processes coincide with the so-called birth-death processes (Karlin and McGregor [15]) which is a stationary Markov process whose transition probability matrix $P_{ij}(t) = Pr(x(t) = j \mid x(0) = i)$ is totally positive in the values i and j for every $t > 0$.

Two concrete illustrations of transition probability functions that arise from suitable diffusion processes are [14]:

(i) Let $L_n^{\alpha}(x)$ be the usual Laguerre polynomial, normalized so that $L_n^{\alpha}(0) = \binom{n+\alpha}{n}$, and let P(t) be the infinite matrix with elements

$$P_{mn}(t) = \int_0^{\infty} e^{-xt} L_n^{\alpha}(x) L_m^{\alpha}(x) x^{\alpha} e^{-x} dx.$$

Then P(t) is strictly TP for each fixed $t > 0$ and $\alpha > -1$. This is an example of a transition probability matrix for a birth-death process.

(ii) The Wiener process on the real line is a strong Markov process with continuous path functions. The direct product of n copies of this process is the n-dimensional Wiener process which is known to be a strong Markov process. Therefore the transition probability function $P(t,x,E) = 1/(\sqrt{4\pi t}) \int_E \exp[-(x-y)^2/4t]dy$ is totally positive for $t > 0$.

5. Applications in Inventory Problem.

Suppose that the probability density f(x) of demand for each period is a PF_3. The policy followed is to maintain the stock size at a fixed level S which will be suitably chosen so as to minimize appropriate expected costs, or is determined by a fixed capacity restriction. At the end of each period an order is placed to replenish the stock consumed during that period so that a constant stock level is maintained on the books. Delivery takes place n periods later. The expected cost for a stationary period as a function of the lag is

$$L(n) = \int_0^S h(S-y) f^{(n)}(y)dy + \int_S^{\infty} \rho(y-S) f^{(n)}(y)dy \qquad (7)$$

where S is fixed, h represents the storage cost function and ρ the penalty cost function.

Let h and ρ be convex increasing functions with $h(0) = \rho(0) = 0$. Then

we may write

$$L(n) = \int r(y) f^{(n)}(y) dy,$$

where

$$r(y) = \begin{cases} h(S-y) & \text{for } 0 \le y \le S \\ p(y-S) & \text{for } S < y. \end{cases} \tag{8}$$

Now $r(y)$ is a convex function. Hence by the convexity preserving property of (4), we conclude that $L(n)$ is a convex function. Thus, if the length of lag increases, the marginal expected loss increases.

Very interesting applications of total positivity are found in system supply problems. Suppose we wish to determine the intitial spare-parts kit for a complex system which provides maximum assurance against system shutdown due to shortage of essential components during a period of length t under a budget for spares C_0. We assume that a failed component is instantly replaced by a spare, if available. Only spares originally provided may be used for replacement, i.e., no resupply of spares can occur during the period. The system contains d_i operating components of type i, i = 1, 2, ..., k. The length of life of the j^{th} operating component of the i^{th} type is assumed to be an independent random variable with PF_k density f_{ij}, j = 1, 2, ..., d_i. The unit cost of a component of type i is c_i.

Our problem is to find n_i, the number of spares initially provided of the i^{th} type, such that $\prod_{i=1}^{k} P_i(n_i)$ is maximized subject to $\sum_{i=1}^{k} n_i c_i \le c_0$ and $n_i = 0, 1, 2, \ldots$ for $i = 1, 2, \ldots, k$, where $P_i(m)$ = probability of experiencing $\le m$ failures of type i.

In Black and Proschan [4], a detailed discussion of methods is given for computing the solution when each $\ell n P_i(m)$ is concave in m, or equivalently, when each $P_i(n-m)$ is a TP_2 sequence in n and m. To show $P_i(n-m)$ is a TP_2 sequence in n and m, we note:

1. $c_{ij}(n)$, the probability of requiring n replacements of operating component i, j, is a PF_2 sequence in n for each fixed i and j.

2. $c_i(n)$, the probability of requiring n replacements of type i, is a PF_2 sequence in n for each i, since $c_i(n) = c_{i1} * c_{i2} * \dots * c_{id_i}(n)$.

3. $P_i(n-m)$ is a TP_2 sequence in n and m for each i, since

(a) $P_i(n) = \sum_{m=-\infty}^{\infty} c(n-m)q(m)$, where

$$q(m) = \begin{cases} 1 & \text{for } m = 0, 1, 2, \dots \\ 0 & \text{otherwise,} \end{cases}$$

(b) $q(m)$ is a PF_∞ sequence,

and

(c) the convolution of PF_k is PF_k.

A procedure for computing the optimal spare parts kit in terms of $P_i(m)$ is given in [4]: For arbitrary $r > 0$, for those i such that $\ell nP_i(1)-\ell nP_i(0) < rc_i$, define $n_i^*(r) = 0$; for the remaining i, define $n_i^*(r)$ as 1 + [largest n such that $\ell nP_i(n+1) - \ell nP_i(n) \geq rc_i$]. Compute $c[n^*(r)] = \sum_{i=1}^{k} c_i n_i^*(r)$.

n^* is optimal when c_0 is once of the values assumed by $c[n^*(r)]$ as r varies over $(0,\infty)$.

6. Applications in Reliability and Life Testing.

A life distribution F is said to have increasing (decreasing) failure rate, denoted by IFR (DFR), if $\log[1-F(t)] \equiv \log \overline{F}(t)$ is concave (convex on $[0,\infty)$). If F has a density f, then the <u>failure rate at time t</u> is defined by $r(t) = f(t)/\overline{F}(t)$ for $F(t) < 1$. Distributions with monotone failure rate are of considerable practical interest and such distributions constitute a very large class.

The monotonicity properties of the failure rate function $r(t)$ are intimately connected with the theory of total positivity. The statement that a distribution F has an increasing failure rate is equivalent to the statement that $\overline{F}(x-y)$ is TP_2 in x and y, or $\overline{F}(x)$ is PF_2.

The concept of TP yields fruitful applications in shock models. We say that a distribution F has increasing failure rate average (IFRA) if $(1/t)[-\log \overline{F}(t)]$ is increasing in $t > 0$, or equivalently, $[\overline{F}(t)]^{1/t}$ is decreasing in $t > 0$. An IFRA distribution provides a natural description of coherent system life when system components are independent IFR. The IFRA distribution also arises naturally when shocks occur randomly according to a Poisson process with intensity λ. The i^{th} shock causes a random amount X_i of damage, where X_1, X_2, ... are independently distributed with common distribution F.

A device fails when the total accumulated damage exceeds a specified capacity or threshold x. Let $\overline{H}(t)$ denote the probability that the device survives $[0,t]$.

Then

$$\overline{H}(t) = \begin{cases} \sum_{k=0}^{\infty} \overline{P}_k e^{-\lambda t}(\lambda t)^k/k! & \text{for } 0 \le t < \infty \\ 1 & \text{for } t < 0. \end{cases} \tag{9}$$

Note that $e^{-\lambda t}[(\lambda t)^k/k!]$ represents the Poisson probability that the device experiences exactly k shocks in $[0,t]$, while $\overline{P}_k = F^{(k)}(x)$ represents the probability that the total damage accumulated over the k shocks does not exceed the threshold x, with $1 = \overline{P}_0 \ge \overline{P}_1 \ge \overline{P}_2 \ge \ldots$.

As key tools in deriving the main result in shock models, the methods of total positivity are employed and in particular the variation diminishing property of TP functions.

If $\bar{P}_k^{1/k}$ is decreasing in k, $\bar{P}_k - \xi$, $0 \le \xi \le 1$, has at most one sign change, from + to - if one occurs. Then it follows from the variation diminishing property that $[\bar{H}(t)]^{1/t}$ is decreasing in t, i.e., H is IFRA.

The following inplications are readily checked:

PF_2 density → IFR distribution → IFRA distribution.

For further discussion and illustrations of the usefulness of total positivity in reliability practices we refer to Barlow and Proschan [3].

7. Multivariate Total Positivity and its Relationship to Qualitative Notions of Dependency.

The following natural generalization of TP_2 was introduced and studied by Karlin and Rinott [18].

Definition. Consider a function $f(\underline{x})$ defined on $X = X_1 \times X_2 \times \ldots \times X_k$ where each X_i is totally ordered. We say that $f(\underline{x})$ is multivariate totally positive of order 2 or MTP_2 if

$$f(\underline{x} \vee \underline{y})\ f\ (\underline{x} \wedge \underline{y}) \ge f(\underline{x})\ f(\underline{y}) \text{ for every } \underline{x},\ \underline{y} \in X, \tag{10}$$

where $\underline{x} \vee \underline{y} = (\max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_n, y_n))$ and $\underline{x} \wedge \underline{y} = (\min(x_1, y_1), \min(x_2, y_2), \ldots, \min(x_n, y_n))$. In order to verify (10) it suffices to show that $f(x) > 0$ is TP_2 in every pair of variables where the remaining variables are held fixed.

Multivariate normal distributions constitute an important class of MTP_2 probability densities. Let $\underline{X}$ follow the density

$$f(\underline{x}) = (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp[-\tfrac{1}{2}(\underline{x}-\underline{\mu})' B(\underline{x}-\underline{\mu})],$$

where $\Sigma^{-1} = B = \| b_{ij} \|_{i,j=1}^{n}$. This density is TP_2 in each pair of arguments and hence MTP_2 if and only if $b_{ij} \le 0$ for all $i \ne j$.

In a great many situations, the random variables of interest are not independent. To appropriately model these situations Esary, Proschan and Walkup [6] introduced the concept of association of random variables. Random variables $X_1, X_2, \ldots, X_n$ are said to be <u>associated</u> if $\mathrm{Cov}(f(\underline{X}), g(\underline{X})) \geq 0$ for all pairs of increasing functions f and g.

It can be shown [18] that if $\underline{X} = (X_1, X_2, \ldots, X_n)$ has a joint MTP_2 density, the $E[\phi(\underline{X})\psi(\underline{X})] \geq E[\phi(\underline{X})]E[\psi(\underline{X})]$ provided ϕ and ψ are simultaneously monotone increasing (or decreasing). Equivalently, $\mathrm{Cov}(\phi(\underline{X}), \psi(\underline{X})) \geq 0$. Thus an MTP_2 random vector $\underline{X}$ consists of associated random variables.

It is well known that the union of independent sets of associated random variables produces an enlarged set of associated random variables. Clearly increasing functions of associated random variables are again associated. It follows that if $\underline{X}$ and $\underline{Y}$ are independent random variables each with associated components, then the components of $\underline{Z} = \underline{X} + \underline{Y}$ is associated. Thus, in particular, if $\underline{X}$ and $\underline{Y}$ both have MTP_2 densities, then association of $(Z_1, Z_2, \ldots, Z_n)$ is retained. However, $\underline{Z}$ need not have a joint MTP_2 density.

A key to many of the results on positive dependence and probabilistic inequalities for the multinormal, multivariate t, and Wishart distributions obtained by Karlin and Rinott [19] is the degree of MTP_2 property inherent in these distributions. Their main theorem delineates a necessary and sufficient condition that the density of $|\underline{X}| = (|X_1|, |X_2|, \ldots, |X_n|)$ where $\underline{X} = (X_1, X_2, \ldots, X_n)$ is governed by $N(\underline{0}, \Sigma)$ be MTP_2 is that there exists a diagonal matrix D with diagonal elements ± 1 such that the off-diagonal elements of $-D\Sigma^{-1}D$ are all nonnegative. For an illustration of the power of this theorem consider $|\underline{X}| = (|X_1|, |X_2|, \ldots, |X_n|)$ possessing a joint MTP_2 density where $\underline{X} \sim N(\underline{0}, \Sigma)$. Define $S_i = \sum_{\nu=1}^{p} X_{i\nu}^2$, $i = 1, 2, \ldots, n$, where $\underline{X}_\nu = (X_{1\nu}, X_{2\nu}, \ldots, X_{n\nu})$, $\nu = 1, 2, \ldots, p$ are i.i.d. random vectors

satisfying the condition of the theorem. The random variables $S_1, S_2, \ldots, S_n$ are associated and have the distribution of the diagonal elements of a random positive definite $n \times n$ matrix S where S follows the Wishart distribution $W_n(p, \Sigma)$ with p degrees of freedom and parameter Σ. It is established in [19] that $P_r(S_1 \geq c_1, S_2 \geq c_2, \ldots, S_n \geq c_n) \geq \prod_{i=1}^{n} P_r(S_i \geq c_i)$ for any positive c_i. For other applications and ramifications of MTP_2, see Karlin and Rinott [18], [19].

Fahmy et al. [7] exploited the concept of MTP_2 to obtain interesting results on assessing the effect of the sample on the posterior distribution in the Bayesian context.

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